

ON THE STRUCTURE GROUP OF A DECOMPOSABLE MODEL SPACE

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ABSTRACT. We study the structure group of a canonical algebraic curvature tensor built from a symmetric bilinear form, and show that in most cases it coincides with the isometry group of the symmetric form from which it is built. Our main result is that the structure group of the direct sum of such canonical algebraic curvature tensors on a decomposable model space must permute the subspaces V_i on which they are defined. For such an algebraic curvature tensor, we show that if the vector space V is a direct sum of subspaces V_1 and V_2 , the corresponding structure group decomposes as well if V_1 and V_2 are invariant of the action of the structure group on V . We determine the freedom one has in permuting these subspaces, and show these subspaces are invariant if $\dim V_1 \neq \dim V_2$ or if the corresponding symmetric forms defined on those subspaces have different (but not reversed) signatures, so that in this situation, only the trivial permutation is allowable. We exhibit a model space that realizes the full permutation group, and, with exception to the balanced signature case, show the corresponding structure group is isomorphic to the wreath product of the structure group of a given symmetric bilinear form by the symmetric group. Using these results, we conclude that the structure group of any member of this family is isomorphic to a direct product of wreath products of pseudo-orthogonal groups by certain subgroups of the symmetric group.

1. INTRODUCTION

Let V be a real vector space of finite dimension N , let $V^* := \text{Hom}(V, \mathbb{R})$ be its dual. An object $R \in \otimes^4 V^*$ is called an *algebraic curvature tensor* if it satisfies the following three properties, the last of which is known as the *Bianchi identity*:

$$(1.a) \quad \begin{aligned} R(x, y, z, w) &= -R(y, x, z, w), \\ R(x, y, z, w) &= R(z, w, x, y), \text{ and} \\ 0 &= R(x, y, z, w) + R(x, z, w, y) \\ &\quad + R(x, w, y, z). \end{aligned}$$

If (M, g) is a pseudo-Riemannian manifold, then one may use the Levi-Civita connection ∇ to compute the Riemann curvature tensor $R^\nabla \in \otimes^4 T^*M$, and the evaluation of this tensor at a point $P \in M$ produces the algebraic curvature tensor $R_P^\nabla \in \otimes^4 T_P^*M$, where T^*M is the cotangent bundle of M , and T_P^*M is the cotangent space of M at P . Moreover, it is a classical differential geometric fact that every algebraic curvature tensor R can be realized as the curvature tensor of a pseudo-Riemannian manifold at a point [9]. Thus it can be said that these

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algebraic curvature tensors are an algebraic portrait of the curvature of a manifold at a point, and an understanding of these algebraic objects often translates into a subsequent understanding of the geometrical object they represent. For example, an understanding of the Osserman conjecture in the higher signature setting is concerned with an algebraic understanding of the Jordan normal form of the Jacobi operator [9], Stanilov-Tsankov theory [1] is concerned with the commutativity of certain other natural operators associated to the Riemann curvature tensor, and other authors have studied certain algebraic questions concerning these algebraic curvature tensors simply because these questions are of interest in their own right. Several examples of this include work on the algebraic properties of the Jacobi operator on complex model spaces [10], the study of the linear independence of certain sets of algebraic curvature tensors [4], and results aimed at improving the efficiency with which one may express a given algebraic curvature tensor [4, 5, 13].

Let $\alpha_1, \dots, \alpha_n$ be a collection of contravariant tensors on V . We call the tuple $\mathfrak{M} := (V, \alpha_1, \dots, \alpha_n)$ a *model space*. For example, if φ is a symmetric bilinear form, and R is an algebraic curvature tensor, (V, φ, R) is a model space. In some cases it is convenient to distinguish certain types of model spaces from others. For example, in [6, 10] the pair (V, R) is referred to as a *weak model space*, although in the current work it is not necessary to make this distinction.

There is a natural action of the general linear group $Gl(V)$ on any contravariant tensor $\alpha \in \otimes^s V^*$. Namely, if $A \in Gl(V)$, we may define $(A^* \alpha)(x_1, \dots, x_s) = \alpha(Ax_1, \dots, Ax_s)$. This differs slightly from the standard representation theoretic approach, since one would normally need to define $A^* \alpha$ by first precomposing with A^{-1} to ensure that $\rho(A)(\alpha) = A^* \alpha$ defines a representation (i.e., that $\rho : Gl(V) \rightarrow \text{End}(\otimes^s V^*)$ is a homomorphism). We define the *structure group* $G_{\mathfrak{M}}$ of a model space $\mathfrak{M} = (V, \alpha_1, \dots, \alpha_n)$ as

$$G_{\mathfrak{M}} = \{A \in Gl(V) \mid A^* \alpha_i = \alpha_i \text{ for } i = 1, \dots, n\}.$$

In the event that $n = 1$ so that the model space $\mathfrak{M} = (V, \alpha)$, then we sometimes write $G_{\mathfrak{M}} = G_{\alpha}$ for simplicity when there is no confusion as to what is meant. In addition, we may also refer to G_{α} as the structure group of α for simplicity, rather than as the structure group of the model space (V, α) .

The concept of a structure group is well-known to mathematicians. For example, if φ is a positive-definite inner product, then $G_{\varphi} = O(N)$, the familiar orthogonal group. If one simply trivially notes that $Gl(V)$ is the structure group of the trivial model space consisting solely of V , then many important quantities are dependent upon the observation that the quantity they compute be “independent of the particular basis chosen,” the determinant and trace of a linear operator, for example.

There are many nontrivial examples where an understanding of a model space’s structure group gives rise to more significant and useful information. A pseudo-Riemannian manifold (M, g) is called *curvature homogeneous* if there is a model space $\mathfrak{M} := (V, \varphi, R)$ and for all $P \in M$ there exists a linear isometry $\phi_P : V \rightarrow T_P M$ with $\phi_P^* R_P^{\nabla} = R$, where φ is an inner product with the same signature as g , and R is an algebraic curvature tensor. It is common to search for a non-constant isometry invariant to determine when a curvature homogeneous manifold is not locally homogeneous, one of the aims in the broad study of curvature homogeneity. See, for example [14] in the Riemannian setting, and [3, 7] in the higher signature setting. The Weyl scalar invariants usually provide such an invariant, and in the Riemannian setting, if they do not, then the manifold is locally homogeneous [15].

It is the case in the higher signature setting that all of these scalar invariants could vanish (Walker metrics are such a family [2, 12]); in this case one needs to look further. It is therefore a common practice to compute the structure group of $\mathfrak{M} = (V, \varphi, R)$, and produce a quantity (using a geometric quantity other than just R) that is invariant under the action of this group. This has been done in [7, 8, 11], and most recently in [6]. Thus, an understanding of $G_{\mathfrak{M}}$ is useful in its own right, and crucial to the study of curvature homogeneity in the higher signature setting.

It is the goal of this paper to compute the structure group of a large and useful family of (weak) model spaces, and to study the effect of the decomposition of certain model spaces on the corresponding structure group. We make this more precise before formally stating these results.

Let $S^2(V)$ be the space of symmetric bilinear forms on V , and let $\mathcal{A}(V)$ be the vector space of algebraic curvature tensors. We define

$$R_{\varphi}(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

It is well known [9] that $R_{\varphi} \in \mathcal{A}(V)$, and that $\mathcal{A}(V) = \text{Span}\{R_{\varphi} | \varphi \in S^2(V)\}$. The R_{φ} have geometrical significance as well: if (M, g) is isometrically embedded in pseudo-Euclidean space $\mathbb{R}^{\dim(M)+\kappa}$, then the curvature tensor of M is of the form $\sum_{i=1}^{\kappa} \pm R_{\varphi_i}$ (see, for example [5]). For these reasons (in particular, when $\kappa = 1$), the tensor R_{φ} is sometimes called a *canonical* algebraic curvature tensor.

There is one final preliminary notion to introduce before we introduce our main results and begin our study. Let $\mathfrak{M} = (V, \alpha_1, \dots, \alpha_n)$ be a model space. Suppose $V = V_1 \oplus V_2$, where $\dim(V_i) \geq 1$. If $\alpha \in \otimes^t V^*$, then we write $V_1 \perp_{\alpha} V_2$ if for $x_i \in V_i$, we have $\alpha(x_1, x_2, v_1, \dots, v_{t-2}) = \alpha(x_2, x_1, v_1, \dots, v_{t-2}) = 0$ for all $v_1, \dots, v_{t-2} \in V$. We say that \mathfrak{M} is *decomposable* if there exist subspaces V_1 and V_2 with $V = V_1 \oplus V_2$, $\dim(V_i) \geq 1$, and $V_1 \perp_{\alpha_s} V_2$ for every $s = 1, \dots, n$. In this event, we write $\alpha_s = \alpha_s^1 \oplus \alpha_s^2$, where α_s^i is the restriction of α_s to V_i , and $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$, where $\mathfrak{M}_i = (V_i, \alpha_1^i, \dots, \alpha_n^i)$. We say that \mathfrak{M} is *indecomposable* if it is not decomposable. For example $R \in \mathcal{A}(V)$, and

$$\ker(R) = \{v \in V | R(v, x, y, z) = 0 \text{ for all } x, y, z \in V\} \neq 0,$$

then there is the decomposition $(V, R) \cong (\bar{V}, \bar{R}) \oplus (\ker(R), 0)$, where $\pi : V \rightarrow V/\ker(R) = \bar{V}$, and \bar{R} is characterized by $\pi^* \bar{R} = R$. It follows that $\ker(\bar{R}) = 0$. We will investigate this situation in Section 2.

The concept of indecomposability is also a familiar one: consider again the situation that $\varphi \in S^2(V)$ is positive definite. One usually refers to a decomposition $(V, \varphi) = (V_1, \varphi_1) \oplus (V_2, \varphi_2)$ simply as $V_1 \perp V_2$. Notice that in the associated structure group $G_{\varphi} = O(N)$, the subspaces V_1 and V_2 are never A -invariant for all $A \in G_{\varphi}$. Also note that $O(N)$ does not decompose as the group theoretic internal direct product $O(\dim(V_1)) \times O(\dim(V_2))$. In fact, since orthonormal bases exist on inner product spaces, one always has the complete decomposition $(V, \varphi) = \oplus_{i=1}^N (V_i, \varphi_i)$, where $\dim(V_i) = 1$. Our main result, Theorem 1.2, offers a very different picture of this situation using the model space $\oplus_{i=1}^k (V_i, R_{\varphi_i})$, where $\varphi_i \in S^2(V)$. See Theorem 1.2 and Section 4 for a description of the effect of this decomposition on its corresponding structure group.

We now give an outline of this paper and list our main results. Section 2 is a short survey of prerequisite information, and gives a general characterization of G_R in the event $\ker(R) \neq \{0\}$, when $R \in \mathcal{A}(V)$. We then give a computation of the structure group $G_{R_{\varphi}}$ when φ is nondegenerate in Theorem 1.1, completing

the characterization of G_{R_φ} in terms of G_φ . Assertions (3) and (4) of Theorem 1.1 are not new results, since Assertion (3) is observational (although see [8] for a nontrivial application), while Assertion (4) is obvious upon consideration of the curvature identities in Equation (1.a); we include them here for completeness.

Theorem 1.1. *Suppose $\varphi \in S^2(V)$ is nondegenerate and has signature (p, q) .*

- (1) *If $\text{Rank}(\varphi) \geq 3$, then $G_{R_\varphi} = \{A \in \text{Gl}(V) \mid A^*\varphi = \pm\varphi\}$.*
- (2) *If $\text{Rank}(\varphi) \geq 3$, and $p \neq q$, then $G_{R_\varphi} = G_\varphi$.*
- (3) *If $\text{Rank}(\varphi) = 2$, then $G_{R_\varphi} = \{A \in \text{Gl}(V) \mid \det(A) = \pm 1\}$.*
- (4) *If $\text{Rank}(\varphi) \leq 1$, then $R_\varphi = 0$, so $G_{R_\varphi} = \text{Gl}(V)$.*

Section 3 is an investigation of G_R , when $R = \bigoplus_{i=1}^k R_{\varphi_i}$. Such an R has geometric significance as the curvature tensor of a certain hypersurface embedding, or as a skew-Tsankov curvature tensor on a Riemannian model space [1]. In this section, we lay out the proof of our main result Theorem 1.2 in the technical Lemma 3.2; with exception to Corollary 1.3, the conclusion of Theorem 1.2 will be the foundation of our subsequent results in this paper. Denote the group of permutations of $\{1, \dots, k\}$ as Sym_k .

Theorem 1.2. *Let $\varphi_i \in S^2(V)$. Suppose $(V, R) = \bigoplus_{i=1}^k (V_i, R_{\varphi_i})$ is a model space with $\ker R = 0$, and let $A \in G_R$. Then there exists $\sigma \in \text{Sym}_k$ so that $A : V_i \rightarrow V_{\sigma(i)}$.*

Section 4 is devoted to the corollaries of Theorem 1.2, and we give a complete description of G_R for $R = \bigoplus_{i=1}^k R_{\varphi_i}$ in terms of the signatures of the defining forms φ_i . Corollary 1.3 demonstrates that if the model space decomposes into subspaces which are invariant under the action of the structure group, then the structure group itself decomposes as a (group theoretic) internal direct product.

Corollary 1.3. *Let $(V, R) = (V_1, R_1) \oplus (V_2, R_2)$, and suppose that V_1 and V_2 are g -invariant for all $g \in G_R$. Then $G_R \cong G_{R_1} \times G_{R_2}$, the group theoretic internal direct product of G_{R_1} and G_{R_2} .*

We use Corollary 1.3 and Theorem 1.2 to show in Corollary 1.4 that it is impossible for any member of the structure group to permute subspaces of different dimension. Corollary 1.4 will also demonstrate that unless the signature of the forms involved are compatible, it is also impossible to permute subspaces of the same dimension. According to Theorem 1.2, there is a well-defined subgroup $\text{Sym}_k^R \leq \text{Sym}_k$ corresponding to every algebraic curvature tensor R of the form found in Theorem 1.2. The following corollary demonstrates that this subgroup $\text{Sym}_k^R \neq \text{Sym}_k$ should the dimension of any of the subspaces V_i or, up to replacing φ with $-\varphi$, the signature of the forms involved differ.

Corollary 1.4. *Let $\varphi_s \in S^2(V_s)$ be nondegenerate for $s = 1, \dots, k$, and suppose $(V, R) = \bigoplus_{s=1}^k (V_s, R_{\varphi_s})$. Let $A \in G_R$, and suppose that for some i, j , the signature of φ_s is (p_s, q_s) for $s = i, j$.*

- (1) *If $A : V_i \rightarrow V_j$, then $\dim(V_i) = \dim(V_j)$.*
- (2) *If $A : V_i \rightarrow V_j$, and $\text{Rank}(\varphi_i) \geq 2$, then $(p_i, q_i) = (p_j, q_j)$ or (q_j, p_j) .*
- (3) *If $(p_i, q_i) = (p_j, q_j)$ or (q_j, p_j) , then there exists a $B \in G_R$ where $B : V_i \rightarrow V_j$.*

We conclude Section 4 with a study of how freely the structure group may permute the subspaces V_i , again, in the situation of Theorem 1.2. Since Corollary

1.4 forbids the exchange of subspaces in certain situations, we consider the situation $V_i \cong V_j = W$, and $\varphi_i = \pm \varphi_j$ for every i and j . Without any loss of generality, we may freely replace φ_j with $-\varphi_j$ if necessary and assume below in Corollary 1.5 that $\varphi = \varphi_j$ for all j .

Corollary 1.5. *Suppose $(V, R) = \oplus_{i=1}^k (W, R_\varphi)$, where $\varphi \in S^2(W)$ is nondegenerate. Then G_R is isomorphic to the wreath product $G_{R_\varphi} \wr \text{Sym}_k$.*

When Corollary 1.5 is used in concert with Corollary 1.3, we arrive at the following result, which easily generalizes from two direct summands $V = \oplus_{p=1}^2 V_p$ to any finite number of direct summands, and completes the classification of the structure group of the algebraic curvature tensors considered here.

Corollary 1.6. *Let $(V_p, R_p) = \oplus_{i=1}^{k_p} (W_p, R_{\varphi_p})$, and let $(V, R) = (V_1, R_1) \oplus (V_2, R_2)$. If $\dim W_1 \neq \dim W_2$, then $G_R \cong (G_{R_{\varphi_1}} \wr \text{Sym}_{k_1}) \times (G_{R_{\varphi_2}} \wr \text{Sym}_{k_2})$.*

We include a short and nontechnical summary of our results in Section 5 that summarizes our work, and gives a method of characterizing any structure group of any algebraic curvature tensor which is the direct sum of any finite number of canonical algebraic curvature tensors.

The authors have a final preliminary remark before we begin our study: we hope this work will also serve as a reference point and learning tool for the further study of structure groups. As a result, many of the descriptions and discussions are meant to be well-referenced, descriptive, and complete. Most of our proofs will use basic methods to keep this work accessible to non-experts.

2. PRELIMINARY NOTIONS AND THE DETERMINATION OF G_{R_φ}

There are several preliminary comments we will need to make that will properly set the stage for our subsequent study of structure groups in general. These preliminary comments are either observational, or can be found in [10]. We conclude this section with a computation of the structure group G_{R_φ} .

Suppose (V, R) is a model space with $R \in \mathcal{A}$. If $x \in \ker R$, and $A \in G_R$, then for $A\tilde{y} = y$, $A\tilde{z} = z$, and $A\tilde{w} = w$, we have

$$R(Ax, \tilde{y}, \tilde{z}, \tilde{w}) = A^*R(x, y, z, w) = 0.$$

Thus $A : \ker R \rightarrow \ker R$, and so $\ker R$ is an invariant subspace. So choosing a basis for $\ker R$ and extending it to a full basis for V demonstrates that every $A \in G_R$ takes the block matrix form with respect to any such basis

$$A = \begin{bmatrix} \bar{A} & 0 \\ C & N \end{bmatrix},$$

where $N \in \text{Gl}(\ker R)$, C is any submatrix of appropriate size, and $\bar{A} \in G_{\bar{R}}$, where $\bar{R} \in \mathcal{A}(\bar{V})$, and $\bar{V} := V/\ker R$, and \bar{R} is the pullback of R under the canonical projection $\pi : V \rightarrow \bar{V}$ as discussed in the introduction. It is for this reason that we study model spaces (V, R) where $\ker R = 0$, since if $\ker R \neq 0$, the computation of G_R translates directly into a study of an associated algebraic curvature tensor with trivial kernel, as in Theorem 1.1. It is known [10] that if $\text{Rank}(\varphi) \geq 2$, then $\ker R_\varphi = \ker \varphi$, so the assumption of nondegeneracy of φ in Assertions (1)–(3) of Theorem 1.1 is equivalent to R_φ having a trivial kernel.

Proof of Theorem 1.1. Suppose $A \in G_{R_\varphi}$. By the definition of R_φ , we have $R_\varphi = A^*R_\varphi = R_{A^*\varphi}$. If $\text{Rank}(\varphi) \geq 3$, we may use Lemma 1.6.3 of [10] to conclude

that $A^*\varphi = \pm\varphi$. Since $R_\varphi = R_{-\varphi}$, the opposite containment holds, and Assertion (1) is established. To prove Assertion (2), we must eliminate the possibility that $A^*\varphi = -\varphi$, i.e., A is a para-isometry. But it is well known that para-isometries only exist in the balanced signature setting. Indeed, such a map would exchange the causal type of any element in an orthonormal basis. If $\text{Rank}(\varphi) = \dim V = 2$, then using the curvature symmetries, one has for $A \in G_{R_\varphi}$ that $R(x, y, y, x)$ is the only nonzero entry up to the symmetries found in Equation (1.a), where $\{x, y\}$ is a linearly independent set. One easily computes

$$R_\varphi(Ax, Ay, Ay, Ax) = (\det A)^2 R_\varphi(x, y, y, x).$$

Since $\text{Rank}(\varphi) = 2$, we have $R_\varphi \neq 0$, so $(\det A)^2 = 1$. Assertion (3) now follows. To prove Assertion (4), we note that $\mathcal{A}(V) = 0$ for any vector space of dimension less than 2. \square

3. THE DETERMINATION OF G_R , WHERE $R = \oplus R_{\varphi_i}$

Now that we have a complete understanding of G_{R_φ} , we turn our attention to pursuing an understanding of G_R where $R = \oplus_{i=1}^k R_{\varphi_i}$, on a model space that is decomposable. According to the discussion preceding the proof of Theorem 1.1, we need only consider those R with $\ker R = 0$, and Lemma 3.1 demonstrates that this is equivalent to each $\varphi_i \in S^2(V)$ being nondegenerate. Moreover, it is shown in [10] that (V, R_φ) is an indecomposable model space if and only if φ is nondegenerate, so that our assumed decomposition is, in a certain sense, a complete one.

In this section we establish our main result regarding the structure group of the decomposable model space $(V, R) = \oplus_{i=1}^k (V_i, R_{\varphi_i})$, with the assumption $\ker R = 0$. We begin with a straightforward lemma.

Lemma 3.1. *Let $(V, R) = (V_1, R_1) \oplus (V_2, R_2)$ be a decomposable model space with $\ker R = 0$. Let $\varphi_1, \varphi_2, \varphi \in S^2(V^*)$.*

- (1) *We have $\ker R_i \cap V_i = 0$, for $i = 1, 2$.*
- (2) *If $R_i = R_{\varphi_i}$, and $\text{Rank}(\varphi_i) \geq 2$ for $i = 1, 2$, then $\ker \varphi_i \cap V_i = 0$.*
- (3) *Suppose $\ker \varphi = 0$, and $\beta = \{e_1, \dots, e_\ell\}$ is an orthonormal basis with respect to φ . Then, for every $e_i \in \beta$ and $j \neq i$, the entries $R_\varphi(e_i, e_j, e_j, e_i) \neq 0$ are the only nonzero entries of R_φ on this basis.*

Proof. We establish Assertion (1) by proving $\ker R_i \cap V_i \subseteq \ker R = 0$. Without loss of generality, suppose $v \in \ker R_1 \cap V_1$. Then since $v \in V_1$ and $R = R_1 \oplus R_2$, for any $x, y, z \in V$ we have $R(v, x, y, z) = R_1(v, \tilde{x}, \tilde{y}, \tilde{z})$, where \tilde{x}, \tilde{y} , and \tilde{z} are the projections of x, y , and z , respectively, to V_1 . Since $v \in \ker R_1$, we have $R(v, x, y, z) = R_1(v, \tilde{x}, \tilde{y}, \tilde{z}) = 0$. Thus $v \in \ker R$, and so $v = 0$.

Assertions (2) and (3) are almost immediate. Since $\ker R = 0$, by Assertion (1), we have $\ker R_{\varphi_i} \cap V_i = 0$. Since $\ker R_{\varphi_i} = \ker \varphi_i$ when $\text{Rank}(\varphi_i) \geq 2$, Assertion (2) follows. Assertion (3) follows as well, since in the event $\ker \varphi = 0$, there are no null vectors in any orthonormal basis, and so

$$R_\varphi(e_i, e_j, e_j, e_i) = \varphi(e_i, e_i)\varphi(e_j, e_j) = \pm 1 \neq 0.$$

In the event that i, j, k are distinct, then one verifies $R_\varphi(e_i, e_j, e_k, e_\ell) = 0$. \square

We may now prove Lemma 3.2, and the proof of our main result Theorem 1.2 will follow as a direct result. We first pause to establish some useful notation. Suppose V_i are subspaces of V with $\dim(V) = N$, and $\oplus_{i=1}^k V_i = V$, and β_i is a basis for

V_i . Now $\beta = \beta_1 \cup \dots \cup \beta_k = \{e_1, \dots, e_N\}$ is an ordered basis for V . Set $a_1 = 1$, and for each $r = 2, \dots, k$, recursively set $a_r = a_{r-1} + \dim(V_{r-1})$, so that e_{a_p} is the first basis vector of $\beta_p = \{e_{a_p}, \dots, e_{a_{p+1}-1}\}$. For each index $i = 1, \dots, N$, we define $n_i \in \{1, \dots, k\}$ to be the unique number so that $e_i \in \beta_{n_i}$. For example, $n_{a_p} = p$. This tool will allow us to locate each basis vector according to the subspace that contains it.

The proof of Lemma 3.2 is somewhat technical, and so we provide a nontechnical summary and short example to demonstrate the method of proof.

Lemma 3.2. *Suppose $(V, R) = \bigoplus_{i=1}^k (V_i, R_{\varphi_i})$ is a model space with $\ker R = 0$, and let $A \in G_R$. Let β_i be an orthonormal basis for V_i with respect to φ_i , and create the ordered basis β for V as above. Let $f_i = Ae_i = \sum_j a_{ji}e_j$.*

- (1) *For each $i = 1, \dots, N$, there exists a well-defined number w_i so that if $n_i \neq n_t$, for any s with $n_{w_i} = n_s$, then $a_{st} = 0$.*
- (2) *For any t with $n_t \neq n_i$, we have $f_t \in \text{Span}(\cup_{j \neq n_{w_i}} \beta_j)$.*
- (3) *The assignment $n_i \mapsto \sigma(i) = n_{w_i}$ is well-defined.*
- (4) *The function $\sigma(n_i) = n_{w_i}$ is a permutation.*
- (5) *$Ae_i \in \text{Span}(\beta_{\sigma(n_i)})$ for all $i \in \{1, \dots, N\}$.*
- (6) *$A : V_i \rightarrow V_{\sigma(i)}$ for all $i \in \{1, \dots, k\}$.*

Proof. Note that the hypotheses forbid $\dim V_i < 2$, since otherwise $\ker R \neq 0$. Since A is nonsingular, $A\beta = \{f_1, \dots, f_N\}$ is a new basis for V . Since $A \in G_R$, we must have

$$R(e_i, e_j, e_j, e_i) = A^*R(e_i, e_j, e_j, e_i) = R(f_i, f_j, f_j, f_i) = \varepsilon_{ij},$$

where

$$\varepsilon_{ij} = \begin{cases} \pm 1 & \text{if } n_i = n_j \\ 0 & \text{if } n_i \neq n_j \end{cases}.$$

By Lemma 3.1, for $s \neq w$ and $n_s = n_w$,

$$(3.a) \quad R(f_r, e_s, f_t, e_w) = \pm a_{wr}a_{st},$$

and if additionally, $n_r \neq n_t$, then by assumption $R(f_r, x, f_t, y) = A^*R(e_r, \tilde{x}, e_t, \tilde{y}) = 0$, where $\tilde{x} = A^{-1}x$, and $\tilde{y} = A^{-1}y$. So we conclude that if $s \neq w$, $n_s = n_w$, and $n_r \neq n_t$, then $R(f_r, e_s, f_t, e_w) = a_{wr}a_{st} = 0$.

We are now ready to establish Assertion (1). Let $i \in \{1, \dots, N\}$ be chosen. Since $Ae_i = \sum_j a_{ji}e_j \neq 0$, there exists some smallest index w so that $a_{wi} \neq 0$. Set $w_i = w$; note that this assignment is well-defined since we choose the smallest such w . Choose any index t with $n_t \neq n_i$. According to Equation (3.a) we conclude that $R(f_i, e_s, f_t, e_w) = \pm a_{wi}a_{st} = 0$ so that $a_{st} = 0$ when $s \neq w$ and $n_s = n_w$. We show presently that $a_{wt} = 0$ for all t with $n_t \neq n_i$, completing the proof of Assertion (1).

If there is a t with $n_t \neq n_i$ with $a_{wt} \neq 0$, then for any a_{cb} with $n_b = n_i$ and $c \neq w$, we have $R(f_t, e_c, f_b, e_w) = \pm a_{wt}a_{cb} = 0$. But now $a_{cb} = 0$ for all b , since we have already considered the case $n_b \neq n_i$ (for $b = t$ above). It follows that $e_b \notin \text{Span}\{f_1, \dots, f_N\}$, a contradiction.

We now prove Assertion (2) as a straightforward consequence of Assertion (1). Fix i , find w_i according to Assertion (1), and choose any t with $n_i \neq n_t$. We have $f_t = Ae_t = \sum_j a_{jt}e_j$. By Assertion (1), for every j with $n_j = n_{w_i}$, we have $a_{jt} = 0$, so that any vectors e_j with $n_j = n_{w_i}$ do not appear in the sum $\sum_j a_{jt}e_j = f_t$. Assertion (2) now follows.

We begin our proof of Assertion (3) with the following observation. According to Assertion (1), we have that if $i \neq j$, then $n_{w_{a_i}} \neq n_{w_{a_j}}$, so that the function $i \mapsto n_{w_{a_i}}$ is injective, and hence bijective. We presently show that $n_i \mapsto n_{w_i}$ is well-defined.

We suppose to the contrary that it is not. This is equivalent to the assertion that there exists indices q and p with $n_q = p = n_{a_p}$, but $n_{w_q} \neq n_{w_{a_p}}$. Since $i \mapsto n_{w_{a_i}}$ is surjective, there exists an index r with $n_{w_{a_r}} = n_{w_q}$ and $a_r \neq a_p$. So $n_{a_r} = r \neq p = n_{a_p} = n_q$, and Assertion (2) forces $f_q \in \text{Span}(\cup_{j \neq n_{w_q}} \beta_j)$; this is a contradiction by the definition of w_q .

Assertion (4) follows easily, since one may freely choose to evaluate σ using a_p instead of any other i with $n_i = p$, and thus σ agrees with the bijection $n_i = p \mapsto n_{w_{a_p}} = n_{w_i}$.

We now prove Assertion (5). Choose any index i . By Assertion (4), for every ℓ with $n_\ell \neq n_{w_i}$ there exists an a_p so that $n_{w_{a_p}} = n_\ell$, and $n_{a_p} \neq n_i$, and so Assertion (1) (using the index $a_p \in \{1, \dots, N\}$) forces the coefficients $a_{i\ell} = 0$ for $n_\ell \neq n_{w_i}$, which shows $Ae_i \in \text{Span}(\beta_{n_{w_i}})$. Assertion (5) follows since, by definition, $n_{w_i} = \sigma(n_i)$. Assertion (6) follows from Assertion (5). \square

We supply a representative example of the situation in Lemma 3.2, since the proof is somewhat dense.

Example 3.3. Suppose that $k = 3$, and each V_i is of dimension 2, so that $V = V_1 \oplus V_2 \oplus V_3$ has dimension 6. Once the basis β is found as in the statement of Lemma 3.2, we may express A in terms of this basis, so that the i th column of A are the components of the vector f_i with respect to the basis β . One divides the matrix up into pieces, according to the numbers n_i . Since $f_1 = \sum_{j=1}^6 a_{j1}e_j$, one of these $a_{j1} \neq 0$. Suppose the smallest w so that $a_{w1} \neq 0$ is when $w = 4$. So we have $w_1 = 4$, so that $n_{w_1} = 2$. This means that $a_{11} = a_{21} = a_{31} = 0$, and $a_{41} \neq 0$. The method of proof of Assertion (1) is to first match the known nonzero entry a_{41} with a_{33}, a_{34}, a_{35} , and a_{36} . These are the a_{st} with $n_t \neq n_1 = 1$, and s with $n_s = n_{w_i} = 2$, but $s = 3 \neq 4 = w_1$. Equation (3.a) shows that these $a_{st} = 0$:

$$A = \left[\begin{array}{cc|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array} \right] = \left[\begin{array}{cc|cc|cc} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline 0 & a_{32} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array} \right].$$

If any of a_{43}, a_{44}, a_{45} or a_{46} are nonzero (now $s = 4 = w_1$), then we could conclude using Equation (3.a) that $a_{32} = 0$ as well, in which case A is singular: a contradiction. So we must have

$$A = \left[\begin{array}{cc|cc|cc} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline 0 & a_{32} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array} \right] = \left[\begin{array}{cc|cc|cc} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline 0 & a_{32} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 & 0 \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array} \right].$$

One can see now that Ae_i is not in the span of e_3 and e_4 for $i = 3, 4, 5, 6$, the conclusion of Assertion (2).

To continue our example, we study $f_3 = Ae_3$. Suppose $a_{13} = a_{23} = 0$, and $a_{53} \neq 0$, so that $w_3 = 5$, $n_{w_3} = 3$, and $Ae_3 = a_{53}e_5 + a_{63}e_6$. Applying Assertion (1) to the lower left and lower right elements of A , we have

$$A = \left[\begin{array}{cc|cc|cc} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline 0 & a_{32} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 & 0 \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array} \right] = \left[\begin{array}{cc|cc|cc} 0 & a_{12} & 0 & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & 0 & a_{24} & a_{25} & a_{26} \\ \hline 0 & a_{32} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_{53} & a_{54} & 0 & 0 \\ 0 & 0 & a_{63} & a_{64} & 0 & 0 \end{array} \right].$$

One sees again that Assertion (2) holds, since Ae_i is not in the span of e_5 and e_6 for $i = 1, 2, 5, 6$.

Finally, it is now clear that either a_{15} or a_{25} is nonzero, so that $n_{w_5} = 1$. A final application of Assertion (1) shows that the upper left and upper center elements of A are all zero:

$$A = \left[\begin{array}{cc|cc|cc} 0 & a_{12} & 0 & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & 0 & a_{24} & a_{25} & a_{26} \\ \hline 0 & a_{32} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_{53} & a_{54} & 0 & 0 \\ 0 & 0 & a_{63} & a_{64} & 0 & 0 \end{array} \right] = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & a_{15} & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ \hline 0 & a_{32} & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_{53} & a_{54} & 0 & 0 \\ 0 & 0 & a_{63} & a_{64} & 0 & 0 \end{array} \right].$$

Once one has done this analysis, the remaining assertions are easy to see. The assignment $p \mapsto n_{w_{a_p}}$ is bijective, since every one of the basis vectors $e_1 = e_{a_1}$, $e_3 = e_{a_2}$, and $e_5 = e_{a_3}$ clearly must satisfy $\{1, 2, 3\} = \{n_{w_1}, n_{w_3}, n_{w_5}\}$. That $n_{w_2} = n_{w_1}$ also follows from this fact: if this were not the case, n_{w_2} would be equal to n_{w_3} or n_{w_5} , and as is evident in our matrix above (as a result of Assertion (1)), this is not possible. Similarly for $n_{w_4} = n_{w_3}$, and $n_{w_6} = n_{w_5}$. So $\sigma(n_i) = n_{w_i}$ is well-defined, and we may freely choose to evaluate $\sigma(n_i) = \sigma(n_{a_p})$ when $n_i = p = n_{a_p}$, and this function is bijective since $\{1, 2, 3\} = \{n_{w_1}, n_{w_3}, n_{w_5}\}$. In fact, one can see from our example that $\sigma = (123)$, and $A : V_i \rightarrow V_{\sigma(i)}$. \square

Theorem 1.2 follows directly from Lemma 3.2. We remark in passing that our method of proof will carry over to any $\alpha \in \otimes^\ell V^*$ that satisfies the relations found in Assertion (3) of Lemma 3.1. Specifically, we do not use the Bianchi identity anywhere in Section 3, and so it may be possible to apply this methodology to other contravariant tensors to obtain similar results. Specifically, there is another construction of algebraic curvature tensors using antisymmetric 2-forms [9, 10], and our method of proof of Lemma 3.2 and Theorem 1.2 apply equally well in that circumstance as well.

4. COROLLARIES OF THEOREM 1.2

This section is devoted entirely to establishing the corollaries of Theorem 1.2. We begin by establishing Corollary 1.3.

Proof of Corollary 1.3. Let β_i be a basis for V_i , and let $\beta = \beta_1 \cup \beta_2$ be an ordered basis for V . Then the hypothesis that each V_i is g -invariant for all $g \in G_R$ implies

that, given any element $g \in G_R$, there exists matrices g_1 and g_2 so that

$$(4.a) \quad [g]_\beta = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}.$$

On this basis, let $j_i : G_{R_i} \rightarrow G_R$ be the inclusion of G_{R_i} into G_R defined as follows:

$$(4.b) \quad j_1(g_1) = \begin{bmatrix} g_1 & 0 \\ 0 & I \end{bmatrix}, \quad \text{and} \quad j_2(g_2) = \begin{bmatrix} I & 0 \\ 0 & g_2 \end{bmatrix}$$

It follows that $G_{R_i} \cong j_i(G_{R_i})$. We complete the proof by showing that G_R is the internal direct product of $j_1(G_{R_1}) \cong G_{R_1}$ and $j_2(G_{R_2}) \cong G_{R_2}$.

First, we note that $j_1(G_{R_1}) \cap j_2(G_{R_2})$ is trivial according to Equations (4.a) and (4.b). Also note that for every $g \in G_R$, we have

$$[g]_\beta = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} = j_1(g_1)j_2(g_2),$$

so that $G_R = j_1(G_{R_1})j_2(G_{R_2})$. Finally, if $A \in j_1(G_{R_1})$ and $g \in G_R$, then expressing the conjugate of A by g as a matrix shows that $g^{-1}Ag \in G_R$ leaves V_1 invariant and is the identity on V_2 . Thus for any $x, y, z, w \in V_1$, we have

$$\begin{aligned} (g^{-1}Ag)^*R_1(x, y, z, w) &= R_1((g^{-1}Ag)x, (g^{-1}Ag)y, (g^{-1}Ag)z, (g^{-1}Ag)w) \\ &= R((g^{-1}Ag)x, (g^{-1}Ag)y, (g^{-1}Ag)z, (g^{-1}Ag)w) \\ &= (g^{-1}Ag)^*R(x, y, z, w) \\ &= R(x, y, z, w) \\ &= R_1(x, y, z, w). \end{aligned}$$

So $g^{-1}Ag \in G_{R_1}$, and $G_{R_1} \trianglelefteq G_R$. Similarly for G_{R_2} . We conclude $G_R = j_1(G_{R_1}) \times j_2(G_{R_2}) \cong G_{R_1} \times G_{R_2}$. \square

The proof of Corollary 1.4 follows mostly from basic linear algebraic observations:

Proof of Corollary 1.4. To prove Assertion (1), suppose $\dim V_i \neq \dim V_j$, and there exists an $A \in G_R$ with $A : V_i \rightarrow V_j$. Since $A^{-1} \in G_R$ as well and $A^{-1} : V_j \rightarrow V_i$, without loss of generality we may assume $\dim(V_i) > \dim(V_j)$ in which case $A|_{V_i} : V_i \rightarrow V_j$ must not have full rank: a contradiction.

We now prove Assertion (2). Suppose $x_i, y_i, z_i, w_i \in V_i$. Then $Ax_i, Ay_i, Az_i, Aw_i \in V_j$, and

$$\begin{aligned} R(x_i, y_i, z_i, w_i) &= R_{\varphi_i}(x_i, y_i, z_i, w_i), \\ R(x_i, y_i, z_i, w_i) &= A^*R(x_i, y_i, z_i, w_i) \\ &= R(Ax_i, Ay_i, Az_i, Aw_i) \\ &= R_{\varphi_j}(Ax_i, Ay_i, Az_i, Aw_i) \\ &= R_{A^*\varphi_j}(x_i, y_i, z_i, w_i). \end{aligned}$$

If $\text{Rank}(\varphi_i) \geq 3$, then $\varphi_i = \pm A^*\varphi_j$ (see [10]), and the result follows in that case. According to our assumptions and Assertion (1), if $\text{Rank}(\varphi_i) = 2$ we know $\dim V_i = \dim V_j = \text{Rank}(\varphi_j) = 2$. If $\{x, y\}$ is a basis for V_i , then the following equation must hold:

$$R_{\varphi_i}(x, y, y, x) = R_{\varphi_j}(Ax, Ay, Ay, Ax) = (\det A)^2 R_{\varphi_j}(x, y, y, x).$$

We note that in dimension 2, the signature of φ_i is determined by the sign of $R_{\varphi_i}(x, y, y, x) = \varphi_i(x, x)\varphi_i(y, y) - \varphi_i(x, y)^2$. Thus, $R_{\varphi_i}(x, y, y, x)$ has the same sign as $R_{\varphi_j}(x, y, y, x)$ if and only if the conclusion of Assertion (2) holds.

To prove the final assertion, we find bases $\beta_s = \{e_1^s, \dots, e_{\dim(V_s)}^s\}$ for V_s which are orthonormal with respect to φ_s . We define $Be_t^s = e_t^s$ for $s \neq i, j$, $Be_t^i = e_t^j$, and $Be_t^j = e_t^i$. \square

Before establishing Corollaries 1.5 and 1.6, we pause to review the wreath product as a group theoretic construction (see page 172 in [16]). Let G be any group. The symmetric group Sym_k acts on $\Pi_{i=1}^k G$ (the direct product of k copies of G) by permuting the indices: $\theta(\sigma)(g_1, \dots, g_k) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(k)})$. Thus, $\theta : Sym_k \rightarrow Aut(\Pi_{i=1}^k G)$ is a homomorphism. The wreath product $G \wr Sym_k$ of G by Sym_k is the semidirect product

$$G \wr Sym_k := (\Pi_{i=1}^k G) \rtimes_{\theta} Sym_k$$

and accordingly has the binary operation given by

$$(h_1, \dots, h_k; \tau)(g_1, \dots, g_k; \sigma) = (h_1 g_{\tau^{-1}(1)}, \dots, h_k g_{\tau^{-1}(k)}; \tau\sigma).$$

Proof of Corollary 1.5. For simplicity, denote G_{R_φ} as G . Define $\Phi : (G \wr Sym_k) \rightarrow G_R$ on the vector $(v_1, \dots, v_k) \in V_1 \oplus \dots \oplus V_k$ as

$$\Phi(g_1, \dots, g_k; \sigma)(v_1, \dots, v_k) = (g_1 v_{\sigma^{-1}(1)}, \dots, g_k v_{\sigma^{-1}(k)}).$$

One observes that $\Phi(g_1, \dots, g_k; \sigma) \in G_R$. In addition, if $(g_1, \dots, g_k; \sigma) \in \ker \Phi$, then we must have $g_i v_{\sigma^{-1}(i)} = v_i$ for all $v_i \in V_i$, so that σ is the identity, and g_i is the identity for all i . So Φ is injective. If $A \in G_R$, then there exists a $\sigma \in Sym_k$ as in Theorem 1.2 with $A : V_i \rightarrow V_{\sigma(i)}$. If we denote $A_i = A|_{V_i} \in G$ and set $g_i = A_{\sigma^{-1}(i)}$, then one has $\Phi(g_1, \dots, g_k; \sigma) = A$. We show Φ preserves the wreath product structure to complete the proof that Φ is an isomorphism:

$$\begin{aligned} & \Phi(h_1, \dots, h_k; \tau) \Phi(g_1, \dots, g_k; \sigma)(v_1, \dots, v_k) \\ &= \Phi(h_1, \dots, h_k; \tau)(g_1 v_{\sigma^{-1}(1)}, \dots, g_k v_{\sigma^{-1}(k)}) \\ &= (h_1 g_{\tau^{-1}(1)} v_{\sigma^{-1}\tau^{-1}(1)}, \dots, h_k g_{\tau^{-1}(k)} v_{\sigma^{-1}\tau^{-1}(k)}) \\ &= (h_1 g_{\tau^{-1}(1)} v_{(\tau\sigma)^{-1}(1)}, \dots, h_k g_{\tau^{-1}(k)} v_{(\tau\sigma)^{-1}(k)}) \\ &= \Phi(h_1 g_{\tau^{-1}(1)}, \dots, h_k g_{\tau^{-1}(k)}; \tau\sigma)(v_1, \dots, v_k) \\ &= \Phi[(h_1, \dots, h_k; \tau)(g_1, \dots, g_k; \sigma)](v_1, \dots, v_k). \end{aligned}$$

□

Proof of Corollary 1.6. Set $V_p = \bigoplus_{i=1}^{k_p} W_p$. Since $\dim(W_1) \neq \dim(W_2)$, we have that each V_p are invariant. Using Corollary 1.3, we identify the structure group G_R as the direct product of the structure groups $G_{R_1} \times G_{R_2}$. Using Corollary 1.5, we identify each $G_{R_p} \cong G_{\varphi_p} \wr Sym_{k_p}$. The result follows.

5. SUMMARY OF RESULTS

This paper aims to understand the structure group of the indecomposable models space of the form (V, R_φ) , or of the decomposable model space (V, R) , where $R = \bigoplus_{i=1}^k R_{\varphi_i}$. According to the discussion preceding the proof of Theorem 1.1, we answer these questions by considering the situation where $\ker R = 0$, which, for the situations considered here, amount to assuming that the symmetric forms involved are all nondegenerate.

Section 2 computes the structure group G_{R_φ} as G_φ unless φ has rank 2, or is of balanced signature. Section 3 contains our main result: to each element of the structure group G_R for $R = \bigoplus_{i=1}^k R_{\varphi_i}$, there is a permutation σ with $A : V_i \rightarrow V_{\sigma(i)}$. We apply these results to various situations of broad interest in Section 4 that help us to classify, up to group isomorphism, the structure group G_R when $R \in \mathcal{A}(V)$ is the direct sum of canonical algebraic curvature tensors. We show that if there are ever two subspaces of V which are invariant by the action of the structure group, then the structure group itself decomposes as an internal direct product.

In this case, the decomposition of the model space gives rise to a decomposition of the structure group. Such a situation arises if, for example if $V = \oplus_{i=1}^k V_i$, the subspaces V_i have different dimensions. In the event the subspaces V_i have the same dimension and the forms φ_i all have the same signature (or reversed signature), then the structure group can be recovered entirely from this data as the wreath product of G_{R_φ} (which has been computed already in Section 2), and the full symmetric group Sym_k . We also show that in the event the subspaces V_i and V_j share the same dimension but $\varphi_i \neq \pm\varphi_j$, then any element of the structure group must *not* permute V_i to V_j . We close our study by noting that combinations of these results are also possible, and these combinations allow one to determine the (group) isomorphism class of G_R . We finish our study by describing how one actually would do this in practice.

Let $(V, R) = \oplus_{i=1}^k (V_i, R_{\varphi_i})$. Since $R_{\varphi_i} = R_{-\varphi_i}$, exchange if necessary φ_i with $-\varphi_i$ to force the signature (p_i, q_i) to satisfy $p_i \leq q_i$. Then partition the V_i according to the signatures of these φ_i defined on each V_i . If one direct sums each (V_i, R_{φ_i}) making up any partition, then the structure group of the resulting model space will be a wreath product of G_φ by some (full) symmetric group, where φ is any one of the forms put in this partition. According to Corollary 1.4, each of the new model spaces on each partition are modeled on vector subspaces which are G_R -invariant. According to Corollary 1.3, the resulting structure group G_R will be isomorphic to a direct product of wreath products.

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